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Bolotin's Method Applied to the Buckling and Lateral Vibration of Stressed Plates

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Introduction

IN a recent Note by King and Lin[1] the edge effect method proposed by Bolotin[2] was applied to the problem of the lateral vibration of rectangular orthotropic and isotropic plates subject to various boundary conditions. The method was shown to be remarkably accurate when used for determining the natural frequencies of such plates. A natural extension of this work is to the buckling and lateral vibration of rectangular plates subject to in-plane forces which are time invariant and constant over the area of the plate and have their principal directions parallel to the plate edges. It is the purpose of this Note to illustrate this extension and to demonstrate its applicability by the presentation of numerical results for a particular plate.

Development of the Analysis

The development of the analysis follows the steps outlined by King and Lin[1] or, alternatively, as presented by Dickinson and Warburton[3] in their paper on the edge effect method applied to plate systems. The plate is assumed to lie in the xy-plane, to be bounded by edges $x = 0, a$ and $y = 0, b$; to be of a uniform thickness, rectangularly orthotropic material having its symmetry axes orthogonal to the plate boundaries and acted upon by constant in-plane forces per unit width N_x and N_y (tensile positive) acting in the x and y directions, respectively.

For free vibration, the lateral displacement $w(x, y)e^{i\omega t}$ is governed by the equation

$$D_x \partial^4 w/\partial x^4 + 2H\partial^4 w/\partial x^2 \partial y^2 + D_y \partial^4 w/\partial y^4 - N_x \partial^2 w/\partial x^2 - N_y \partial^2 w/\partial y^2 - \rho\omega^2 w = 0 \qquad (1)$$

where D_x, H, D_y are plate flexural rigidities and ρ is the plate mass/unit area. Considering the effects of the edges to be localized, the displacement away from the edges may be represented by $w = f(x)g(y) = W_0 \sin k_x(x/a - \alpha) \sin k_y(y/b - \beta)$ which satisfies Eq. (1) provided that

$$\omega^2 = (1/\rho)[D_x(k_x/a)^4 + 2H(k_x k_y/ab)^2 + D_y(k_y/b)^4 + N_x(k_x/a)^2 + N_y(k_y/b)^2] \qquad (2)$$

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Near to an edge an exponential term is included in $f(x)$ [or $g(y)$], which decays away from the edge. Thus, in the vicinity of $x = 0$, an additional term $Ae^{\gamma_x x/a}$ is included in $f(x)$ and near $y = 0$ a term $Be^{\gamma_y y/b}$ in $g(y)$. For these additional terms to be admissible by Eq. (1) and for γ to be real and negative, it is necessary that

$$\gamma_x = -[k_x^2 + 2(H/D_x)(k_y a/b)^2 + N_x a^2/D_x]^{1/2}$$

and

$$\gamma_y = -[k_y^2 + 2(H/D_y)(k_x b/a)^2 + N_y b^2/D_y]^{1/2}$$

Near to the edge $x = 0$, $f(x)$ may be written

$$f(x) = A_1 \sin k_x x/a + A_2 \cos k_x x/a + A_3 e^{\gamma_x x/a}$$

and near to edge $x = a$,

$$f(x) = A_4 \sin k_x(1 - x/a) + A_5 \cos k_x(1 - x/a) + A_6 e^{\gamma_x(1 - x/a)}$$

Similar expressions may be written for $g(y)$ near $y = 0$ and $y = b$, involving six constants B_1 to B_6.

Continuity of displacement functions away from the plate edges leads to two equations giving A_4 and A_5 in terms of A_1 and A_2 and two similar equations giving B_4 and B_5 in terms of B_1 and B_2. The 12 arbitrary constants A_s and B_s are thus eliminated by consideration of the eight plate boundary conditions and the four continuity conditions, resulting in two simultaneous equations in k_x and k_y, the solution of which enables ω^2 to be determined from Eq. (2).

In the case of buckling, $\omega = 0$; thus, if either N_x or N_y is known, or if the relationship between N_x and N_y is known, the appropriate buckling values are readily obtainable.

Numerical Results

In order to demonstrate the application of the edge effect method to plates subject to in-plane forces, the method was used to determine the buckling loads and fundamental natural frequencies of a square, clamped plate subject to hydrostatic in-plane force ($N_x = N_y = N$), for various conditions of orthotropy. Corresponding, accurate natural frequencies had previously been determined by the author using a series solution[4] and were thus available for comparison.

The simultaneous equations which result for the clamped plate are

$$2 \cos k_x + (\gamma_x/k_x - k_x/\gamma_x) \sin k_x = 0$$

and

$$2 \cos k_y + (\gamma_y/k_y - k_y/\gamma_y) \sin k_y = 0$$

the solutions of which were obtained using a numerical interpretation of Taylor's theorem.

Table 1 shows the fundamental frequency parameters computed using the edge effect method and the series solution and quite close agreement may be seen to be achieved. From the nature of the edge effect method, as discussed in Refs. 1 and 3,

Table 1 Fundamental frequency parameter $\omega ab(\rho/H)^{1/2}$ for a square clamped orthotropic plate subject to hydrostatic in-plane force (tension positive)

		D_x/H					
		1/2		1		2	
Na^2/π^2H	D_y/H	Edge effect	Series[4]	Edge effect	Series[4]	Edge effect	Series[4]
−2		15.78	17.74				
0	1/2	27.47	28.07				
10		54.93	54.98				
20		71.89	71.91				
−2		22.04	23.78	26.73	28.57		
0	1	31.56	32.27	35.09	35.99		
10		57.42	57.50	59.80	59.93		
20		74.04	74.07	76.13	76.17		
−2		31.20	32.66	34.59	36.30	40.90	42.64
0	2	38.54	39.29	41.42	42.40	46.83	47.96
10		61.80	61.92	64.00	64.18	67.91	68.17
20		77.71	77.75	79.70	79.77	83.10	83.21

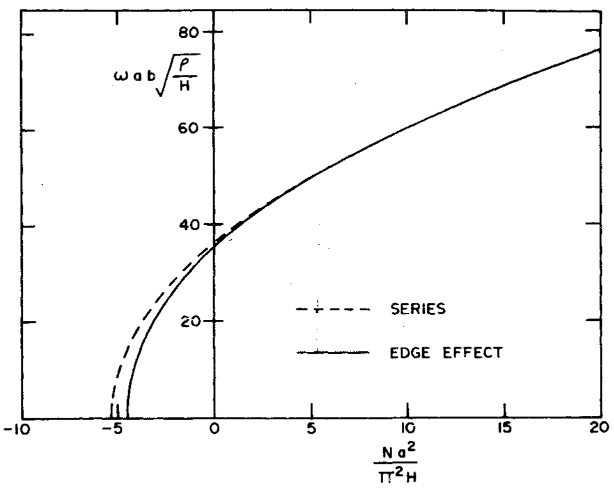

Fig. 1 Graph of fundamental frequency parameter vs in-plane load for square isotropic plate.

Table 2 Hydrostatic buckling parameter $Na^2/\pi^2 D$ for a square clamped orthotropic plate (compression positive)

	D_x/H		
D_y/H	1/2	1	2
1/2	2.91		
1	3.72	4.50 (5.31)[a]	
2	5.20	6.07	7.76

[a] ( ) accurate result by Taylor.[5]

it may be recognized that the closer a plate deflection comes to a sine wave, the smaller the edge effect error and thus the greater the accuracy achieved. This is borne out by the results of Table 1 where it may be seen that the larger discrepancies occur for the compressive forces, where the plate deflection tends to deviate from a sine wave, than for tensile forces, where the plate tends to behave more and more like a membrane, the exact shape for which is a pure sine wave. Figure 1, which shows the variation in frequency with in-plane load for the isotropic plate (computed using the edge effect method and the series solution), further illustrates this trend.

For completeness, Table 2 shows the buckling loads obtained using the edge effect method for the various cases of orthotropy and, in the case of the isotropic plate, shows the accurate value presented by Taylor.[5] The accuracy of the method for buckling problems is somewhat poorer than that for natural frequency calculations for plates subject to zero or tensile in-plane forces. But, since the method is believed to always yield a lower bound for single plates (which is not necessarily the case for plate systems[3]), the results still have utility.

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Separation-Like Similarity Solutions on Two-Dimensional Moving Walls

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Introduction

THE objective of the following Note is to present the results of a study of two-dimensional separation-like velocity profiles on moving walls based on similarity solutions of the incompressible boundary-layer equations. The underlying motivation is the possibility of using similarity solutions as a family of velocity profiles in an integral theory for the nonsimilar boundary-layer development on moving walls. In such a method it is important to identify the profile characteristics associated with separation.

Prandtl[1] defined the important features of an incipient separation velocity profile on a fixed wall in his historic 1904 paper. Separation is preceded by vanishing wall shear stress so that the criterion for separation is $\partial u/\partial y = 0$, $y = 0$.

The Falkner-Skan group of similarity solutions contains a profile which meets this criterion and when the similarity profiles and potential flow pressure distribution are employed in an integral theory separation is predicted at 105.2° on a circular cylinder. This compares with 104.5° predicted by numerical techniques.[2]

The boundary-layer on a moving wall is dominated by the conditions near the surface which prevents the fixed wall form of separation. Application of Prandtl's criterion results in a layer of fluid moving with the wall and not separation. Moore[3] modified Prandtl's criterion as follows:‡

$$\partial u/\partial y = 0, \qquad u = 0 \tag{1}$$

at a coincident point which includes Prandtl's criterion as a special case. When the wall is moving in the same direction as the boundary-layer edge velocity, Moore's criterion requires that the fluid be brought to rest by the adverse pressure gradient at an intermediate point in the boundary layer. Downstream of this point is an imbedded region of reverse flow.

Solutions to the similarity equations which meet this criterion were obtained by Moore for two wall velocities. Additional examples have been computed so that the variation of β, the pressure gradient parameter, and other integral shape factors for this class of separation profiles can be defined.

Equations

The incompressible, two-dimensional boundary-layer equations are transformed into the similarity equation for the

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‡ N. Rott and W. Sears also reached the same conclusion in unpublished work.